

# SECOND YAMABE CONSTANT ON RIEMANNIAN PRODUCTS

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**ABSTRACT.** Let  $(M^m, g)$  be a closed Riemannian manifold ( $m \geq 2$ ) of positive scalar curvature and  $(N^n, h)$  any closed manifold. We study the asymptotic behaviour of the second Yamabe constant and the second  $N$ -Yamabe constant of  $(M \times N, g + th)$  as  $t$  goes to  $+\infty$ . We obtain that  $\lim_{t \rightarrow +\infty} Y^2(M \times N, [g + th]) = 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e])$ . If  $n \geq 2$ , we show the existence of nodal solutions of the Yamabe equation on  $(M \times N, g + th)$  (provided  $t$  large enough). When  $s_g$  is constant, we prove that  $\lim_{t \rightarrow +\infty} Y_N^2(M \times N, g + th) = 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e)$ . Also we study the second Yamabe invariant and the second  $N$ -Yamabe invariant.

**Keywords:** Second Yamabe constant; Yamabe equation; Nodal solutions.

## 1. INTRODUCTION

Let  $(W^k, G)$  be a closed Riemannian manifold of dimension  $k \geq 3$  with scalar curvature  $s_G$ . The Yamabe functional  $J : C^\infty(W) - \{0\} \rightarrow \mathbb{R}$  is defined by

$$J(u) := \frac{\int_W a_k |\nabla u|_G^2 + s_G u^2 dv_G}{\|u\|_{p_k}^2}.$$

where  $a_k := 4(k-1)/(k-2)$  and  $p_k := 2k/(k-2)$ .

The infimum of the Yamabe functional over the set of smooth functions of  $W$ , excluding the zero function, is a conformal invariant and it is called the Yamabe constant of  $W$  in the conformal class of  $G$  (which we are going to denote by  $[G]$ ):

$$Y(W, [G]) = \inf_{u \in C^\infty(W) - \{0\}} J(u).$$

Recall that the conformal Laplacian operator of  $(W, G)$  is

$$L_G := a_k \Delta_G + s_G,$$

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where  $\Delta_G$  is the negative Laplacian, i.e.,  $\Delta_{g_e} u = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$  in the Euclidean space  $(\mathbb{R}^n, g_e)$ .

The celebrated Yamabe problem states that in any conformal class of a closed Riemannian manifold (of dimension at least 3) there exists a Riemannian metric with constant scalar curvature. This was proved in a series of articles by Yamabe [26], Trudinger [25], Aubin [5], and Schoen [22]. Actually, they proved that the Yamabe constant is attained by a smooth positive function  $u_{min}$ . It can be seen that a function  $u_{cp}$  is a critical point of the Yamabe functional if and only if it solves the so called Yamabe equation

$$(1) \quad L_G(u_{cp}) = \lambda |u_{cp}|^{p_k-2} u_{cp}$$

for  $\lambda = J(u_{cp}) / \|u_{cp}\|_{p_k}^{p_k-2}$ . Recall that if  $\tilde{G}$  belongs to  $[G]$ , then

$$L_G(u) = s_{\tilde{G}} u^{p_k-1}$$

where  $u$  is the positive smooth function that satisfies  $\tilde{G} = u^{p_k-2} G$ . Therefore,  $G_{u_{min}} := u_{min}^{p_k-2} G$  must be a metric of constant scalar curvature.

The solution of the Yamabe problem provides a positive smooth solution of the Yamabe equation. Actually, as we pointed out, there is a one to one relationship between the Riemannian metrics with constant scalar curvature in  $[G]$  and positive solutions of the Yamabe equation.

Nevertheless, in order to understand the set of solutions of the Yamabe equation, it seems important to study the nodal solutions, i.e., a sign changing solution of (1). In the last years several authors have addressed the question about the existence and multiplicity of nodal solutions of the Yamabe equation: Hebey and Vaugon [11], Holcman [12], Jourdain [13], Djadli and Jourdain [8], Ammann and Humbert [2], Petean [18], El Sayed [9] among others.

Let

$$\lambda_1(L_G) < \lambda_2(L_G) \leq \lambda_3(L_G) \leq \dots \nearrow +\infty$$

be the sequence of eigenvalues of  $L_G$ , where each eigenvalue appears repeated according to its multiplicity. It is well known that it is an increasing sequence that tends to infinity.

When  $Y(W, [G]) \geq 0$ , it is not difficult to see that

$$Y(W, [G]) = \inf_{\tilde{G} \in [G]} \lambda_1(L_{\tilde{G}}) \text{vol}(W, \tilde{G})^{\frac{2}{k}},$$

where  $\text{vol}(W, \tilde{G})$  is the volume of  $(W, \tilde{G})$ .

In [2], Ammann and Humbert introduced the  $l$ th Yamabe constant. This constant is defined by

$$Y^l(W, [G]) := \inf_{\tilde{G} \in [G]} \lambda_l(L_{\tilde{G}}) \text{vol}(W, \tilde{G})^{\frac{2}{k}}.$$

Like the Yamabe constant, the  $l$ th Yamabe constant is a conformal invariant.

They showed that the second Yamabe constant of a connected Riemannian manifold with non-negative Yamabe constant is never achieved by a Riemannian metric. Nevertheless, if we enlarge the conformal class, allowing generalized metrics (i.e., metrics of the form  $u^{p_k-2}G$  with  $u \in L^{p_k}(W)$ ,  $u \geq 0$ , and  $u$  does not vanish identically), under some assumptions on  $(W, G)$ , the second Yamabe constant is achieved ([2], Corollary 1.7). Moreover, if  $Y^2(W, G) > 0$ , they proved that if a generalized metric  $\tilde{G}$  realizes the second Yamabe constant, then it is of the form  $|w|^{p_k-2}G$  with  $w \in C^{3,\alpha}(W)$  a nodal solution of the Yamabe equation. If  $Y^2(W, G) = 0$  and is attained, then any eigenfunction corresponding to the second eigenvalue of  $L_G$  is a nodal solution.

Therefore, if we know that the second Yamabe constant is achieved, we have a nodal solution of the Yamabe equation. However, this is not the general situation. There exist some Riemannian manifolds for which the second Yamabe constant is not achieved, even by a generalized metric. For instance,  $(S^k, g_0^k)$  where  $g_0^k$  is the round metric of curvature 1 (cf. [2], Proposition 5.3).

Let  $(M, g)$  and  $(N, h)$  be closed Riemannian manifolds and consider the Riemannian product  $(M \times N, g + h)$ . We define the  $N$ -Yamabe constant as the infimum of the Yamabe functional over the set of smooth functions, excluding the zero function, that depend only on  $N$ :

$$Y_N(M \times N, g + h) := \inf_{u \in C^\infty(N) - \{0\}} J(u).$$

Clearly,  $Y(M \times N, g + h) \leq Y_N(M \times N, g + h)$ . The  $N$ -Yamabe constant is not a conformal invariant, but it is scale invariant. It was first introduced by Akutagawa, Florit, and Petean in [1], where they studied, among other things, its behavior on Riemannian products of the form  $(M \times N, g + th)$  with  $t > 0$ .

Actually, the infimum of  $J$  over  $C^\infty(N) - \{0\}$  is a minimum, and it is achieved by a positive smooth function.

When the scalar curvature of the product is constant, the critical points of the Yamabe functional restricted to  $C^\infty(N) - \{0\}$ , satisfy the Yamabe equation, and thereby, also satisfy the subcritical Yamabe equation (recall that  $p_{m+n} < p_n$ ). Hence, if  $Y_N(M \times N, g + h) = J(u)$ , then the metric  $G = u^{p_{m+n}-2}(g + h) \in [g + h]$  has constant scalar curvature. When  $s_{g+h} \leq 0$ , the Yamabe constant of  $(M \times N, g + h)$  is nonpositive, and in this situation, there is essentially only one metric of constant scalar curvature, the metric  $g + h$ . Therefore, this case it is not interesting.

It seems important to consider the  $N$ -Yamabe constant because in some cases the minimizer (or some minimizers) of the Yamabe functional depends only on one of the variables of the product. For instance, it was proved by Kobayashi in [15] and Schoen in [23] that the minimizer of the Yamabe functional on  $(S^n \times S^1, g_0^n + tg_0^1)$  depends only on  $S^1$ . Also, this might be the case for  $(S^n \times \mathbb{H}^m, g_0^n + tg_h)$  (for small values of  $t$ ), where  $(\mathbb{H}^m, g_h)$  is the  $m$ -dimensional Hyperbolic space of curvature  $-1$ . These Riemannian products are interesting, because their Yamabe constants appear in the surgery formula for the Yamabe invariant (see the definition below) proved by Ammann, Dahl, and Humbert in [3].

We define the  $l$ th  $N$ -Yamabe constant as:

$$Y_N^l(M \times N, g + h) := \inf_{G \in [g+h]_N} \lambda_l^N(L_G) \text{vol}(M \times N, G)^{\frac{2}{m+n}},$$

where  $[g + h]_N$  is the set of Riemannian metrics in the conformal class  $[g + h]$  that can be written as  $u^{p_{m+n}-2}(g + h)$ , with  $u$  a positive smooth function that depends only on  $N$ , and  $\lambda_l^N(L_G)$  is the  $l$ th eigenvalue of  $L_G$  restricted to functions that depend only on the variable  $N$ .

A generalized metric  $G = u^{p_{m+n}-2}(g + h)$  is called a generalized  $N$ -metric if  $u$  depends only on  $N$ .

Petean proved ([18], Theorem 1.1) that the second  $N$ -Yamabe constant of a Riemannian product of closed manifolds with constant and positive scalar curvature is always attained by a generalized  $N$ -metric of the form  $|w|^{p_{m+n}-2}(g + h)$  where  $w \in C^{3,\alpha}(N)$  is a nodal solution of the Yamabe equation.

The aim of the present article is study the behaviour of the second Yamabe constant and the second  $N$ -Yamabe constant of a Riemannian product  $(M \times N, g + th)$  with  $t > 0$ . We prove the following results:

**Theorem 1.1.** *Let  $(M^m, g)$  be a closed manifold ( $m \geq 2$ ) with positive scalar curvature and let  $(N^n, h)$  be a closed manifold. Then,*

$$\lim_{t \rightarrow +\infty} Y^2(M \times N, [g + th]) = 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]).$$

From this theorem, as well as from some results in [1] and [2], we obtain:

**Corollary 1.2.** *Let  $(M^m, g)$  as above and let  $(N^n, h)$  be a closed Riemannian manifold ( $n \geq 2$ ). For  $t$  large enough,  $Y^2(M \times N, [g + th])$  is attained by a generalized metric of the form  $|v|^{p_{m+n}-2}(g + th)$ , where  $v$  is a nodal solution of the Yamabe equation on  $(M \times N, g + th)$ . Moreover,  $v \in C^{3,\alpha}(M \times N)$  and is smooth in  $M \times N - \{v^{-1}(0)\}$ .*

We point out that the nodal solutions provided by Corollary 1.2, in general, are not the same solutions provided by ([18], Theorem 1.1), which depend only on  $N$  (see Subsection 3.1 and Remark 3.7).

For the second  $N$ -Yamabe constant we obtain the next theorem:

**Theorem 1.3.** *Let  $(M^m, g)$  be a closed manifold ( $m \geq 2$ ) of positive and constant scalar curvature and  $(N^n, h)$  be any closed manifold. Then,*

$$\lim_{t \rightarrow +\infty} Y_N^2(M \times N, g + th) = 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e).$$

In Subsection 3.3 we will define the second Yamabe constant and the  $N$ -second Yamabe constant for a non-compact manifold. There we prove:

**Theorem 1.4.** *Let  $(M^m, g)$  be a closed manifold of positive scalar curvature. Then,*

$$Y^2(M \times \mathbb{R}^n, g + g_e) = 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]).$$

*If in addition  $(M^m, g)$  has constant scalar curvature, then*

$$Y_{\mathbb{R}^n}^2(M \times \mathbb{R}^n, g + g_e) = 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e).$$

The Yamabe invariant of  $W$ , which we denote by  $Y(W)$ , is the supremum of the Yamabe constants over the set  $\mathcal{M}_W$  of Riemannian metrics on  $W$ :

$$Y(W) := \sup_{G \in \mathcal{M}_W} Y(W, [G]).$$

This important differential invariant was introduced by Kobayashi in [15] and Schoen in [22]. It provides information about the capability of  $W$  to admit a Riemannian metric of positive scalar curvature. More precisely, the Yamabe invariant is positive if and only if the manifold admits a metric of positive scalar curvature.

Similarly, we define the  $l$ th Yamabe invariant of  $W$  by

$$Y^l(W) := \sup_{G \in \mathcal{M}_W} Y^l(W, [G]).$$

For a product  $M \times N$ , we define the  $l$ th  $N$ -Yamabe invariant as

$$Y_N^l(M \times N) := \sup_{g \in \mathcal{M}_M^Y, h \in \mathcal{M}_N} Y_N(M \times N, g + h),$$

where  $\mathcal{M}_M^Y$  is the subset of Yamabe metrics of  $\mathcal{M}_M$ , i.e., metrics that realize the Yamabe constant. By a result due to Pollack [21] we know that for any Riemannian manifold of dimension  $n \geq 3$  with positive Yamabe invariant there exist metrics with a constant scalar curvature  $n(n-1)$  and arbitrarily large volume. Therefore, if we take the supremum among  $\mathcal{M}_M$  instead of  $\mathcal{M}_M^Y$ ,  $Y_N^l(M \times N)$  would be infinite (see the variational characterization of the  $l$ th  $N$ -Yamabe constant in Section 2).

In Section 4, we point out several facts about the second Yamabe invariant and the second  $N$ -Yamabe invariant. Also, taking into account some known bounds for the Yamabe invariant, we show lower bounds for these invariants.

Note that frequently in the literature, the Yamabe constant and the Yamabe invariant are called Yamabe invariant and  $\sigma$ -invariant, respectively. Something similar happens for the  $l$ th Yamabe invariant and for the  $l$ th Yamabe constant. In this article we are not going to use these denominations.

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## 2. PRELIMINARIES

### 2.1. Notation.

Let  $(W^k, G)$  be a Riemannian manifold. Throughout this article we will denote with  $C_{\geq 0}^\infty(W)$  and  $L_{\geq 0}^p(W)$  the set of non-negative functions on  $W$ , excluding the zero function, that belong to  $C^\infty(W)$  and  $L^p(W)$ , respectively. We are going to

denote with  $C_{\geq 0}^\infty(W)$  the positive functions of  $C_{\geq 0}^\infty(W)$ .  $L_{\geq 0, c}^p(W)$  and  $C_{\geq 0, c}^\infty(W)$  will be the subset of functions with compact support that belong to  $L_{\geq 0}^p(W)$  and  $C_{\geq 0}^\infty(W)$ , respectively.

Let  $H$  be one of these spaces of functions:  $C^\infty(W)$ ,  $C_c^\infty(W)$  or  $H_1^2(W)$ . We write  $Gr^l(H)$  for the set of all  $l$ -dimensional subspaces of  $H$ . If  $u \in H$ , we denote with  $Gr_u^l(H)$  the elements of  $Gr^l(H)$  that satisfy: If  $V = \text{span}(v_1, \dots, v_l)$ , then  $\tilde{V} = \text{span}(u^{p_k-2}v_1, \dots, u^{p_k-2}v_l)$  belongs to  $Gr^l(H)$ .

## 2.2. Results from the literature.

Here, for the convenience of the reader, we state some important results from the literature that we are going to use in the next sections.

The following theorem is due to Ammann and Humbert ([2], Theorem 5.4 and Proposition 5.6):

**Theorem 2.1.** *Let  $(W^k, G)$  be a closed Riemannian manifold ( $k \geq 3$ ) with non-negative Yamabe constant. Then,*

$$2^{\frac{2}{k}} Y(W, [G]) \leq Y^2(W, [G]) \leq [Y(W, [G])^{\frac{k}{2}} + Y(S^k)^{\frac{k}{2}}]^{\frac{2}{k}}.$$

*Moreover, if  $Y^2(W, [G])$  is attained and  $W$  is connected, then the left hand side inequality is strict.*

We summarise the main results of [2] (Theorem 1.4, 1.5, and 1.6) in the next theorem:

**Theorem 2.2.** *Assume the same hypothesis as in the theorem above:*

- a)  $Y^2(W, [G])$  is attained by a generalized metric if

$$Y^2(W, [G]) < [Y(W, [G])^{\frac{k}{2}} + Y(S^k)^{\frac{k}{2}}]^{\frac{2}{k}}.$$

*Furthermore, if  $Y^2(W, [G]) > 0$  this generalized metric is of the form  $|w|^{p-2}G$  with  $w \in C^{3,\alpha}(W)$  a nodal solution of the Yamabe equation.*

- b) *The inequality in a) is fulfilled by any non locally conformally flat manifold with  $Y(W, [G]) > 0$  and  $k \geq 11$  or  $Y(W, [G]) = 0$  and  $k \geq 9$ .*

In [1], Akutagawa, Florit, and Petean studied the behavior of the Yamabe constant and the  $N$ -Yamabe constant on Riemannian products. More precisely, they proved the following important result ([1], Theorem 1.1):

**Theorem 2.3.** *Let  $(M^m, g)$  and  $(N^n, h)$  be closed Riemannian manifolds. In addition, assume that  $(M, g)$  is of positive scalar curvature and  $m \geq 2$ . Then,*

$$\lim_{t \rightarrow +\infty} Y(M \times N, [g + th]) = Y(M \times \mathbb{R}^n, [g + g_e]),$$

and

$$\lim_{t \rightarrow +\infty} Y_N(M \times N, g + th) = Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e).$$

If  $(M, g)$  is a closed manifold, then  $(M \times \mathbb{R}^n, g + g_e)$  is complete, with positive injective radius and bounded geometry. Hence, the Sobolev embedding theorem holds (cf. [10], Theorem 3.2). If we assume that the scalar curvature  $s_g$  is positive, then it is not difficult to see that  $Y(M^m \times \mathbb{R}^n, g + g_e) > 0$  (see Section 2.3 for the definition of the Yamabe constant in the non-compact case). If  $m, n \geq 2$ , it was proved in ([1], Theorem 1.3) that

$$(2) \quad 0 < Y(M \times \mathbb{R}^n, [g + g_e]) < Y(S^{m+n}).$$

### 2.3. Yamabe constant on non-compact manifolds.

Note that in the definition of the Yamabe constant the infimum of the Yamabe functional could be taken as well over  $C_{>0}^\infty(W)$ ,  $C_c^\infty(W) - \{0\}$  or  $H_1^2(W) - \{0\}$  and it does not change. Thus, it seems natural (cf. [24]) to define the Yamabe constant of a non-compact manifold  $(W^k, G)$  as

$$Y(W, [G]) := \inf_{u \in C_c^\infty(W) - \{0\}} \frac{\int_W a_k |\nabla u|_G^2 + s_G u^2 dv_G}{\|u\|_{p_k}^2}.$$

The Yamabe constant, also in the noncompact setting, is always bounded from above by the Yamabe constant of  $(S^n, g_0^n)$ . Since  $Y(S^k, [g_0^k]) = Y(S^k)$ , we have that  $Y(W) \leq Y(S^k)$ .

### 2.4. Variational characterization of the $l$ th Yamabe constant.

It is well known the min-max characterization of the  $l$ th eigenvalue of conformal Laplacian of a closed manifold  $(W^k, G)$ :

$$\begin{aligned} \lambda_l(L_G) &= \inf_{V \in Gr^l(C^\infty(W))} \sup_{v \in V - \{0\}} \frac{\int_W L_G(v) v dv_G}{\|v\|_2^2} \\ &= \inf_{V \in Gr^l(H_1^2(W))} \sup_{v \in V - \{0\}} \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\|v\|_2^2}. \end{aligned}$$

For any Riemannian metric  $G_u := u^{p_k-2}G$  in  $[G]$ , the conformal Laplacian satisfies the invariance property

$$L_{G_u}(v) = u^{1-p_k} L_G(uv).$$

Since  $\text{vol}(W, G_u) = \int_W u^{p_k} dv_G$ , we get

$$\begin{aligned} \lambda_l(L_{G_u}) \text{vol}(W, G_u)^{\frac{2}{k}} &= \inf_{V \in Gr^l(H_1^2(W))} \sup_{v \in V - \{0\}} \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\int_W u^{p_k-2} v^2 dv_G} \\ &\quad \times \left( \int_W u^{p_k} dv_G \right)^{\frac{2}{k}}. \end{aligned}$$

Therefore, we have the following characterization of the  $l$ th Yamabe constant of  $(W, G)$ :

$$Y^l(W, [G]) = \inf_{\substack{u \in C_{>0}^\infty(W) \\ V \in Gr^l(H_1^2(W))}} \sup_{v \in V - \{0\}} \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\int_W u^{p_k-2} v^2 dv_G} \left( \int_W u^{p_k} dv_G \right)^{\frac{2}{k}}.$$

If we enlarge the conformal class of  $G$ , allowing generalized metrics, then we obtain

$$Y^l(W, [G]) = \inf_{\substack{u \in L_{>0}^{p_k}(W) \\ V \in Gr_u^l(H_1^2(W))}} \sup_{v \in V - \{0\}} \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\int_W u^{p_k-2} v^2 dv_G} \left( \int_W u^{p_k} dv_G \right)^{\frac{2}{k}}.$$

Let  $(M^m \times N^n, g + h)$  be a Riemannian product of closed manifolds with  $s_g$  constant. If we consider generalized  $N$ -metrics instead of  $N$ -metrics in  $[g + h]$ , we have the following variational characterization of the  $l$ th  $N$ -Yamabe constant:

$$\begin{aligned} Y_N^l(M \times N, g + h) &= \inf_{\substack{u \in L_{>0}^{p_{m+n}}(N) \\ V \in Gr_u^l(H_1^2(N))}} \sup_{v \in V - \{0\}} \frac{\int_N a_k |\nabla v|_{g+h}^2 + s_{g+h} v^2 dv_{g+h}}{\int_N u^{p_{m+n}-2} v^2 dv_{g+h}} \\ &\quad \times \left( \int_N u^{p_{m+n}} dv_{g+h} \right)^{\frac{2}{m+n}} (vol(M, g))^{\frac{2}{m+n}}. \end{aligned}$$

### 3. SECOND YAMABE CONSTANT AND SECOND $N$ -YAMABE CONSTANT ON RIEMANNIAN PRODUCTS

#### 3.1. Second Yamabe constant.

Let  $(M^m, g)$  be a closed manifold ( $m \geq 2$ ) of positive scalar curvature, and  $(N^n, h)$  any closed Riemannian manifold. Note that  $Y(M \times N, [g + th])$  is positive for  $t$  large enough. By Theorem 2.1, we get

$$2^{\frac{2}{k}} Y(M \times N, [g + th]) \leq Y^2(M \times N, [g + th]) \leq [Y(M \times N, [g + th])^{\frac{k}{2}} + Y(S^k)^{\frac{k}{2}}]^{\frac{2}{k}},$$

where  $k = m + n$ . Applying Theorem 2.3 to these inequalities, we obtain the following lemma:

**Lemma 3.1.** *Let  $(M^m, g)$  be a closed manifold ( $m \geq 2$ ) of positive scalar curvature and let  $(N^n, h)$  be any closed manifold. Then,*

$$2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]) \leq \liminf_{t \rightarrow +\infty} Y^2(M \times N, [g + th])$$

and

$$\limsup_{t \rightarrow +\infty} Y^2(M \times N, [g + th]) \leq [Y(M \times \mathbb{R}^n, [g + g_e])^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}]^{\frac{2}{m+n}}.$$



When  $(M, g)$  is  $(S^{m-1}, g_0^{m-1})$  with  $m \geq 3$  and  $(N, h)$  is  $(S^1, g_0^1)$  the lemma above implies that

$$\lim_{t \rightarrow +\infty} Y^2(S^{m-1} \times S^1, g_0^{m-1} + tg_0^1) = 2^{\frac{2}{m}} Y(S^m).$$

Here, we used that  $Y(S^{m-1} \times \mathbb{R}, g_0^{m-1} + g_e) = Y(S^m)$ . But, by the inequality (2), this is no longer true for  $(S^{m-1} \times \mathbb{R}^n, g_0^{m-1} + g_e)$  when  $n \geq 2$ .

*Proof of Theorem 1.1.* From Lemma 3.1 we only have to prove that

$$\limsup_{t \rightarrow +\infty} Y^2(M \times N, [g + th]) \leq 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]).$$

Given  $\varepsilon > 0$ , let  $f = f_\varepsilon \in C_{\geq 0}^\infty(M \times \mathbb{R}^n)$  such that

$$J(f) \leq Y(M \times \mathbb{R}^n, [g + g_e]) + \varepsilon.$$

Assume that the support of  $f$  is included in  $M \times B_R(0)$ , where  $B_R(0)$  is the Euclidean ball centred at 0 with radius  $R$ .

For  $q \in N$ , we denote with  $\exp_q^h$  the exponential map at  $q$  with respect to the metric  $h$  and with  $B_\delta^h(0_q)$  the ball of radius  $\delta$  centred at  $0_q \in T_q N$ .

Let  $q_1$  and  $q_2$  be two points on  $N$ , and consider their normal neighbourhoods  $U_1 = \exp_{q_1}^h(B_\delta^h(0_{q_1}))$  and  $U_2 = \exp_{q_2}^h(B_\delta^h(0_{q_2}))$ . We are going to choose  $\delta > 0$ , such that  $U_1$  and  $U_2$  are disjoint sets and for any normal coordinate system  $x = (x_1, \dots, x_n)$ , we have

$$(1 + \varepsilon)^{-1} dv_{g_e} \leq dv_h \leq (1 + \varepsilon) dv_{g_e}.$$

Note that for the metric  $t^2 h$ , we have  $B_\delta^h(0_{q_i}) = B_{t\delta}^{t^2 h}(0_{q_i})$ . Therefore, if we consider a normal coordinate system  $y = (y_1, \dots, y_n)$  with respect to the metric  $t^2 h$ , we get

$$(1 + \varepsilon)^{-1} dv_{g_e} \leq dv_{t^2 h} \leq (1 + \varepsilon) dv_{g_e}$$

in  $B_{t\delta}^{t^2 h}(0_{q_i})$ .

Let  $t_1$  be such that  $t_1 \delta > R$ . For  $t \geq t_1$ , we are going to identify  $B_{t\delta}(0) \subseteq \mathbb{R}^n$  with  $U_i = \exp_{q_i}^{t^2 h}(B_{t\delta}^{t^2 h}(0_{q_i}))$ . Hence,

$$M \times B_R(0) \subseteq M \times B_{t\delta}(0) \simeq M \times U_i \subseteq M \times N.$$

Let  $\phi_i : M \times N \rightarrow \mathbb{R}$  defined by

$$\phi_i(p, q) := \begin{cases} f(p, q) & (p, q) \in M \times U_i, \\ 0 & (p, q) \notin M \times U_i, \end{cases}$$

and let us consider  $\phi : M \times N \rightarrow \mathbb{R}$  given by

$$\phi := \phi_1 + \phi_2.$$

Clearly,  $\phi_i \in C_{\geq 0}^\infty(M \times N)$ ,  $\phi \in L_{\geq 0}^{p_{m+n}}(M \times N)$ , and the subspace  $V_0 := \text{span}(\phi_1, \phi_2)$  belongs to  $Gr_\phi^2(M \times N)$ .

If we choose  $t_2$  such that  $s_{g+th} \leq (1 + \epsilon)s_g$  for  $t \geq t_2$ , then taking  $t \geq t_3 := \max(t_1^2, t_2)$ , it is not difficult to see that

$$(3) \quad \begin{aligned} & \int_{M \times U_i} a_{m+n} |\nabla \phi_i|_{g+th}^2 + s_{g+th} \phi_i^2 dv_{g+th} \\ & \leq (1 + \epsilon)^3 \int_{M \times B_R(0)} a_{m+n} |\nabla \phi_i|_{g+g_e}^2 + s_g \phi_i^2 dv_{g+g_e}, \end{aligned}$$

and

$$(4) \quad \int_{M \times B_R(0)} \phi_i^{p_{m+n}} dv_{g+g_e} \leq (1 + \epsilon) \int_{M \times U_i} \phi_i^{p_{m+n}} dv_{g+th}.$$

By the variational characterization of the second Yamabe constant we get

$$Y^2(M \times N, [g + th])$$

$$\begin{aligned} & \leq \sup_{v \in V_0 - \{0\}} \frac{\int_{M \times N} a_{m+n} |\nabla v|_{g+th}^2 + s_{g+th} v^2 dv_{g+th}}{\int_{M \times N} \phi^{p_{m+n}-2} v^2 dv_{g+th}} \\ & \quad \times \left( \int_{M \times N} \phi^{p_{m+n}} dv_{g+th} \right)^{\frac{2}{m+n}} \\ & = \sup_{(\alpha_1, \alpha_2) \in \mathbb{R}^2 - \{0\}} \frac{\sum_{i=1}^2 \alpha_i^2 (\int_{M \times N} a_{m+n} |\nabla \phi_i|_{g+th}^2 + s_{g+th} \phi_i^2 dv_{g+th})}{\int_{M \times N} \alpha_1^2 \phi_1^{p_{m+n}} + \alpha_2^2 \phi_2^{p_{m+n}} dv_{g+th}} \\ & \quad \times \left( \int_{M \times N} \phi_1^{p_{m+n}} + \phi_2^{p_{m+n}} dv_{g+th} \right)^{\frac{2}{m+n}} \\ & = 2^{\frac{2}{m+n}} \sup_{(\alpha_1, \alpha_2) \in \mathbb{R}^2 - \{0\}} \frac{\sum_{i=1}^2 \alpha_i^2 (\int_{M \times N} a_{m+n} |\nabla \phi_i|_{g+th}^2 + s_{g+th} \phi_i^2 dv_{g+th})}{(\alpha_1^2 + \alpha_2^2) \|\phi_1\|_{p_{m+n}}^2} \end{aligned}$$

In the last equality, we used that  $\|\phi_1\|_{p_{m+n}} = \|\phi_2\|_{p_{m+n}}$ . Applying the inequality (4), we obtain

$$\begin{aligned} & Y^2(M \times N, [g + th]) \leq 2^{\frac{2}{m+n}} (1 + \epsilon)^{\frac{2}{p_{m+n}}} \\ & \quad \times \sup_{(\alpha_1, \alpha_2) \in \mathbb{R}^2 - \{0\}} \frac{\sum_{i=1}^2 \alpha_i^2 (\int_{M \times N} a_{m+n} |\nabla \phi_i|_{g+th}^2 + s_{g+th} \phi_i^2 dv_{g+th})}{(\alpha_1^2 + \alpha_2^2) (\int_{M \times B_R(0)} \phi_1^{p_{m+n}} dv_{g+g_e})^{\frac{2}{p_{m+n}}}}. \end{aligned}$$

By inequality (3), for any  $t \geq t_3$ , we have

$$Y^2(M \times N, [g + th]) \leq (1 + \epsilon)^{\frac{4(m+n)-2}{m+n}} 2^{\frac{2}{m+n}}$$

$$\begin{aligned} & \times \frac{\int_{M \times B_R(0)} a_{m+n} |\nabla f|_{g+g_e}^2 + s_g f^2 dv_{g+g_e}}{(\int_{M \times B_R(0)} f^{p_{m+n}} dv_{g+g_e})^{\frac{2}{p_{m+n}}}} = (1 + \epsilon)^{\frac{4(m+n)-2}{m+n}} 2^{\frac{2}{m+n}} J(f) \end{aligned}$$

$$\leq (1 + \varepsilon)^{\frac{4(m+n)-2}{m+n}} 2^{\frac{2}{m+n}} (Y(M \times \mathbb{R}^n, g + g_e) + \varepsilon).$$

Finally, letting  $\varepsilon$  go to 0, we obtain that

$$\limsup_{t \rightarrow \infty} Y^2(M \times N, [g + th]) \leq 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]),$$

which finishes the proof.  $\square$

**Remark 3.2.** *The same proof can be adapted to prove that*

$$\limsup_{t \rightarrow +\infty} Y^l(M \times N, [g + th]) \leq l^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]),$$

for  $l \geq 2$ .

**Corollary 3.3.** *Let  $(M^m, g)$  be a closed manifold ( $m \geq 2$ ) with positive scalar curvature and let  $(N^n, h)$  be any closed manifold ( $n \geq 2$ ). Then, for  $t$  large enough, we have*

$$Y^2(M \times N, [g + th]) < [Y(M \times N, [g + th])^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}]^{\frac{2}{m+n}}.$$

*Proof.* Since  $Y(M \times \mathbb{R}^n, [g + g_e]) < Y(S^{m+n})$ , it follows that

$$2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]) < [Y(M \times \mathbb{R}^n, [g + g_e])^{\frac{m+n}{2}} + Y(S^{m+n})^{\frac{m+n}{2}}]^{\frac{2}{m+n}}.$$

On the other hand, we know by Theorem 2.3 that  $\lim_{t \rightarrow +\infty} Y(M \times N, [g + th]) = Y(M \times \mathbb{R}^n, [g + g_e])$ . Thereby, provided  $t$  large enough, Theorem 1.1 implies the desired inequality.  $\square$

Now, Corollary 1.2 is an immediate consequence of the corollary above and Theorem 2.2. Hence, for  $t$  large enough, we have a sign changing solution  $v \in C^{3,\alpha}(M \times N)$  of the equation

$$L_{g+th}v = \lambda|v|^{p_{m+n}-2}v.$$

We can choose  $v$  such that  $\lambda = Y^2(M \times N, [g + th])$ .

Note that in general  $(M \times N, g + th)$  is not locally conformally flat (eventually it is when  $(M, g)$  and  $(N, th)$  have constant sectional curvature 1 and  $-1$ ). Therefore, when  $m + n \geq 11$ , Corollary 1.2 is a direct consequence of Theorem 2.2.

Actually, as we mentioned in the Introduction, the second  $N$ -Yamabe constant of a product  $(M \times N, g + th)$  is attained (when  $s_g$  or  $s_{g+h}$  is constant) by a generalized  $N$ -metric, and this provides a nodal solution of the Yamabe equation on  $(M \times N, g + th)$  that only depends on  $N$ , i.e., a nodal solution of

$$L_{g+h}(w) = Y_N^2(M \times N, g + h)|w|^{p_{m+n}-2}w.$$

However, in general, this solution is not the same solution that the one provided by Corollary 1.2. The reason is that  $Y^2(M \times N, [g + th])$ , generally, will be smaller than  $Y_N^2(M \times N, g + th)$  (see Remark 3.7).

### 3.2. Second $N$ -Yamabe constant.

The second  $N$ -Yamabe constant is always attained by a generalized metric. It can be proved, with the same argument used in [18], that the  $l$ th  $N$ -Yamabe constant is also attained by a generalized metric.

**Lemma 3.4.** *Let  $(M, g)$  and  $(N, h)$  be closed Riemannian manifolds such that  $s_g$  is constant and  $Y_N(M \times N, g + h) \geq 0$ . Then,*

$$2^{\frac{2}{m+n}} Y_N(M \times N, g + h) \leq Y_N^2(M \times N, g + h).$$

The argument to prove the lemma is similar to the one used to prove the first inequality in Theorem 2.1 (for the details see the proof of Proposition 5.6 in [2]). In this situation we only have to restrict to functions that depend only on the  $N$  variable. For convenience of the reader we briefly sketch the proof:

*Proof.* For  $u \in L^{p_{m+n}}(N)$  and  $v \in H_1^2(N) - \{0\}$ , let us consider

$$F_N(u, v) = \frac{\left( \int_N a_{m+n} |\nabla v|_h^2 + s_{g+h} v^2 dv_h \right) \left( \int_N u^{p_{m+n}} dv_h \right)^{\frac{2}{m+n}} \text{vol}(M, g)^{\frac{2}{m+n}}}{\int_N u^{p_{m+n}-2} v^2 dv_h}.$$

The lemma will follow if we prove that for any  $u \in C_{>0}^\infty(N)$ , with  $\|u\|_{p_{m+n}} = 1$ , and any  $V \in Gr^2(C^\infty(N))$  we have

$$(5) \quad \sup_{v \in V - \{0\}} F_N(u, v) \geq 2^{\frac{2}{m+n}} Y_N(M \times N, g + h).$$

The operator  $L_{u^{p_{m+n}-2}(g+h)}$  restricted to  $H_1^2(N)$  has a discrete spectrum

$$0 < \lambda_1^N(L_{u^{p_{m+n}-2}(g+h)}) \leq \lambda_2^N(L_{u^{p_{m+n}-2}(g+h)}) \leq \dots$$

Let  $w_1$  and  $w_2$  be the first two eigenvectors associated with  $\lambda_1^N(L_{u^{p_{m+n}-2}(g+h)})$  and  $\lambda_2^N(L_{u^{p_{m+n}-2}(g+h)})$ , respectively. By the conformal invariance of the conformal Laplacian operator,  $v_1 = uw_1$  and  $v_2 = uw_2$  satisfy

$$L_{g+h}(v_1) = \lambda_1^N(L_{u^{p_{m+n}-2}(g+h)}) u^{p_{m+n}-2} v_1$$

and

$$L_{g+h}(v_2) = \lambda_2^N(L_{u^{p_{m+n}-2}(g+h)}) u^{p_{m+n}-2} v_2.$$

We can choose  $w_1$  and  $w_2$  such that

$$\int_N u^{p_{m+n}} v_1 v_2 dv_h = 0.$$

Notice that by the maximum principle we can also choose  $v_1 > 0$ , then  $v_2$  must change sign.

The supreme (5) in any subspace  $V \in Gr^2(C^\infty(N))$  is greater or equal than  $\sup_{v \in V_0 - \{0\}} F_N(u, v)$  when  $V_0 := \text{span}(v_1, v_2)$ . Actually, we have that

$$\sup_{v \in V_0 - \{0\}} F_N(u, v) = \lambda_2^N(L_{u^{p_{m+n}-2}(g+h)}).$$

Now, using the Hölder inequality and the definition of the  $N$ -Yamabe constant we get

$$2Y_N(M \times N, g + h) \leq \lambda_2^N(L_{u^{p_{m+n}}(g+h)}) \left[ \left( \int_{\{v_2 \geq 0\}} u^{p_{m+n}-2} dv_h \right)^{\frac{p_{m+n}-2}{p_{m+n}}} + \left( \int_{\{v_2 < 0\}} u^{p_{m+n}} dv_h \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \right].$$

Applying again the Hölder inequality, we obtain

$$\left( \int_{\{v_2 \geq 0\}} u^{p_{m+n}} dv_h \right)^{\frac{p_{m+n}-2}{p_{m+n}}} + \left( \int_{\{v_2 < 0\}} u^{p_{m+n}} dv_h \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \leq 2^{\frac{2}{p_{m+n}}}.$$

Therefore,

$$2^{\frac{2}{m+n}} Y_N(M \times N, g + h) \leq \lambda_2^N(L_{u^{p_{m+n}-2}(g+h)}).$$

□

*Proof of Theorem 1.3.* By the positiveness of the scalar curvature of  $(M, g)$ , for any  $t > 0$  we have

$$0 < Y(M \times N, [g + th]) \leq Y_N(M \times N, g + th).$$

Hence, by Lemma 3.4 we have

$$2^{\frac{2}{m+n}} Y_N(M \times N, g + th) \leq Y_N^2(M \times N, g + th).$$

From Theorem 2.3, we obtain

$$2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e) \leq \liminf_{t \rightarrow +\infty} Y_N^2(M \times N, g + th).$$

For any  $\varepsilon > 0$ , we choose  $f = f_\varepsilon \in C_{\geq 0, c}^\infty(\mathbb{R}^n)$  that satisfies

$$J(f) \leq Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e) + \varepsilon,$$

then, it can be proved by a similar argument to the one used in the proof of Theorem 1.1 that

$$\limsup_{t \rightarrow +\infty} Y_N^2(M \times N, g + th) \leq 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e).$$

This completes the proof.

□

**Remark 3.5.** Let  $(M^m, g)$  be a closed Riemannian manifold ( $m \geq 2$ ) of constant positive scalar curvature. Hence by Theorem 2.3 we have that  $Y(M \times \mathbb{R}^n, [g + g_e]) = Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e)$  if and only if

$$\lim_{t \rightarrow +\infty} Y_N(M \times N, g + th) = \lim_{t \rightarrow +\infty} Y(M \times N, [g + th]).$$

for any closed Riemannian manifold  $(N, h)$ . By Theorem 1.1 and Theorem 1.3, the equality  $Y(M \times \mathbb{R}^n, [g + g_e]) = Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e)$  is also equivalent to have

$$\lim_{t \rightarrow +\infty} Y_N^2(M \times N, g + th) = \lim_{t \rightarrow +\infty} Y^2(M \times N, [g + th]),$$

for any closed Riemannian manifold  $(N, h)$ .

For  $m$  and  $n$  positive integers, the  $\alpha_{m,n}$  Gagliardo-Nirenberg constant is defined as

$$\alpha_{m,n} := \left[ \inf_{u \in H_1^2(\mathbb{R}^n) - \{0\}} \frac{(\int_{\mathbb{R}^n} |\nabla u|^2 dv_{g_e})^{\frac{n}{m+n}} (\int_{\mathbb{R}^n} u^2 dv_{g_e})^{\frac{m}{m+n}}}{(\int_{\mathbb{R}^n} |u|^{p_{m+n}} dv_{g_e})^{\frac{m+n-2}{m+n}}} \right]^{-1}.$$

These constants are positive and can be computed numerically. In [1], they were computed for some cases ( $m+n \leq 9$ , with  $n, m \geq 2$ ). Also it was proved in ([1], Theorem 1.4) that for any closed Riemannian manifold  $(M, g)$  of positive constant scalar curvature and with unit volume, it holds

$$(6) \quad Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e) = \frac{A_{m,n} s_g^{\frac{m}{m+n}}}{\alpha_{m,n}},$$

where  $A_{m,n} := (a_{m+n})^{\frac{n}{m+n}} (m+n) m^{-\frac{m}{m+n}} n^{-\frac{n}{m+n}}$ .

An immediate consequence of (6) is:

**Corollary 3.6.** *Let  $(M^m, g)$  be a closed manifold ( $m \geq 2$ ) of constant positive scalar curvature and  $(N^n, h)$  any closed Riemannian manifold. Then,*

$$\lim_{t \rightarrow +\infty} Y_N^2(M \times N, g + th) = \frac{2^{\frac{2}{m+n}} A_{m,n} s_g^{\frac{m}{m+n}} \text{vol}(M, g)^{\frac{2}{m+n}}}{\alpha_{m,n}}.$$

**Remark 3.7.** *If  $(W, G_s) = (M^m \times N^n, s^{-n}g + s^mh)$  where  $(M, g)$  and  $(N, h)$  are closed manifolds of constant positive scalar curvature and unit volume, then  $(W, G_s)$  has constant positive scalar curvature and unit volume too. Nevertheless, the scalar curvature of  $(W, G_s)$  tends to infinity as  $s$  goes to infinity. Therefore, for  $s$  large enough, from (6) we obtain that  $Y(S^{m+n+k}) < Y_{\mathbb{R}^k}(W \times \mathbb{R}^k, G_s + g_e)$ , hence  $Y(W \times \mathbb{R}^k, [G_s + g_e]) < Y_{\mathbb{R}^k}(W \times \mathbb{R}^k, G_s + g_e)$ . This implies that, for any closed  $k$ -dimensional manifold  $(Z, w)$  and  $t$  sufficiently large, we have*

$$Y(W \times Z, [G_s + tw]) < Y_Z(W \times Z, G_s + tw),$$

and

$$Y^2(W \times Z, [G_s + tw]) < Y_Z^2(W \times Z, G_s + tw).$$

### 3.3. Second Yamabe and second $N$ -Yamabe constant on non-compact manifolds.

Throughout this section,  $(W^k, G)$  will be a complete Riemannian manifold, not necessary compact, with  $Y(W, [G]) > 0$ . We define the  $l$ th Yamabe constant of  $(W, G)$  as

$$Y^l(W, G) := \inf_{\substack{u \in L_{\geq 0, c}^{p_k}(W) \\ V \in Gr_u^l(C_c^\infty(W))}} \sup_{v \in V - \{0\}} \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\int_W u^{p_k-2} v^2 dv_G} \left( \int_W u^{p_k} dv_G \right)^{\frac{2}{k}}.$$

**Proposition 3.8.** For  $l \geq 2$ ,  $0 < Y(W, G) = Y^1(W, G) \leq Y^l(W, G)$ .

*Proof.* To prove that  $Y(W, G) \leq Y^l(W, G)$  for  $l \geq 1$ , it is sufficient to show that

$$(7) \quad Y(W, G) \leq \sup_{v \in V - \{0\}} \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\int_W u^{p_k-2} v^2 dv_G}$$

for any  $u \in L_{\geq 0, c}^{p_k}(W)$  with  $\|u\|_{p_k} = 1$  and  $V \in Gr_u^l(C_c^\infty(W))$ .

If  $v \in V - \{0\}$ , by the Hölder inequality, we have that

$$0 < \int_W u^{p_k-2} v^2 dv_G \leq \left( \int_W v^{p_k} dv_G \right)^{\frac{2}{p_k}}.$$

Since  $Y(W, [G]) > 0$ , we have that  $\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G > 0$  for any  $v \in V - \{0\}$ . Thereby, we obtain

$$J(v) = \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\left( \int_W v^{p_k} dv_G \right)^{\frac{2}{p_k}}} \leq \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\int_W u^{p_k-2} v^2 dv_G}.$$

Now, taking supreme on the right hand side of the last inequality we get (7).

Let  $u_i \in C_{\geq 0, c}^\infty(W)$  be a minimizing sequence of  $Y(W, [G])$ . We can assume that  $\|u_i\|_{p_k} = 1$ . Then,

$$\begin{aligned} Y(W, [G]) &\leq Y^1(W, G) \leq \inf_{\substack{V \in Gr_{u_i}^1(C_c^\infty(W)) \\ v \in V - \{0\}}} \frac{\int_W a_k |\nabla v|_G^2 + s_G v^2 dv_G}{\int_W u_i^{p_k-2} v^2 dv_G} \\ &\leq \int_W a_k |\nabla u_i|_G^2 + s_G u_i^2 dv_G = J(u_i) \xrightarrow{i \rightarrow +\infty} Y(W, [G]), \end{aligned}$$

which finishes the proof.  $\square$

Let  $(M^m, g)$  be a closed Riemannian manifold of constant scalar curvature and let  $(N^n, h)$  be a non-compact Riemannian manifold such that  $Y(M \times N, [g+h]) > 0$ . Then, we define the  $l$ th  $N$ -Yamabe constant of  $(M \times N, g+h)$  as

$$\begin{aligned} Y_N^l(M \times N, g+h) &:= \inf_{\substack{u \in L_{\geq 0, c}^{p_{m+n}}(N) \\ V \in Gr_u^l(C_c^\infty(N))}} \sup_{v \in V - \{0\}} \frac{\int_N a_{m+n} |\nabla v|_h^2 + s_{g+h} v^2 dv_h}{\int_N u^{p_{m+n}-2} v^2 dv_h} \\ &\quad \times \left( \int_N u^{p_{m+n}} dv_G \right)^{\frac{2}{m+n}} \left( \text{vol}(M, g) \right)^{\frac{2}{m+n}}. \end{aligned}$$

*Proof of Theorem 1.4.* We are going to prove the statement of the theorem for the second Yamabe constant case. The argument to show the assertion for the second  $N$ -Yamabe constant is similar. We only have to restrict to functions that depend only on  $\mathbb{R}^n$ .

The proof essentially follows along the lines as that of Theorem 4.1 in [2].

First we are going to show that

$$Y^2(M \times \mathbb{R}^n, g + g_e) \leq 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]).$$

Let  $\varepsilon > 0$  and consider  $f = f_\varepsilon \in C_{\geq 0, c}^\infty(M \times \mathbb{R}^n)$  such that

$$J(f) \leq Y(M \times \mathbb{R}^n, [g + g_e]) + \varepsilon.$$

Assume that the support of  $f$  is in  $M \times B_R(0)$ . For  $\tilde{R} > 2R$ , we can choose  $q_1$  and  $q_2$  in  $B_{\tilde{R}}(0)$  such that  $B_R(q_1) \cap B_R(q_2) = \emptyset$  and  $M \times B_R(q_1) \cup M \times B_R(q_2) \subset M \times B_{\tilde{R}}(0)$ . Consider the function  $u := v_1 + v_2$  where  $v_i(p, q) = f(p, q - q_i)$ , and let  $V_0 := \text{span}(v_1, v_2) \in Gr_u^2(C_c^\infty(M \times \mathbb{R}^n))$ . Then,

$$\begin{aligned} Y^2(M \times \mathbb{R}^n, g + g_e) &\leq \sup_{v \in V_0 - \{0\}} \frac{\int_{M \times B_{\tilde{R}}(0)} a_{m+n} |\nabla v|_{g+g_e}^2 + s_g v^2 dv_{g+g_e}}{\int_{M \times B_{\tilde{R}}(0)} u^{p_{m+n}-2} v^2 dv_{g+g_e}} \\ &\quad \times \left( \int_{M \times B_{\tilde{R}}(0)} u^{p_{m+n}} dv_{g+g_e} \right)^{\frac{2}{m+n}} \\ &\leq 2^{\frac{2}{m+n}} J(f) \leq 2^{\frac{2}{m+n}} \left( Y(M \times \mathbb{R}^n, [g + g_e]) + \varepsilon \right). \end{aligned}$$

Letting  $\varepsilon$  go to 0, we obtain the desired inequality.

Let  $u \in L_{\geq 0, c}^{p_{m+n}}(M \times \mathbb{R}^n)$  and  $V \in Gr_u^2(C_c^\infty(M \times \mathbb{R}^n))$ . Let us consider  $F(u, V)$  given by

$$F(u, V) := \sup_{v \in V - \{0\}} \frac{\int_{M \times \mathbb{R}^n} a_{m+n} |\nabla v|_{g+g_e}^2 + s_g v^2 dv_{g+g_e}}{\int_{M \times \mathbb{R}^n} u^{p_{m+n}-2} v^2 dv_{g+g_e}} \left( \int_{M \times \mathbb{R}^n} u^{p_{m+n}} dv_{g+g_e} \right)^{\frac{2}{m+n}}.$$

Since  $H : L_{\geq 0, c}^{p_{m+n}}(M \times \mathbb{R}^n) \times C_c^\infty(M \times \mathbb{R}^n) - \{0\} \rightarrow \mathbb{R}$  defined by

$$H(u, v) := \frac{\int_{M \times \mathbb{R}^n} a_{m+n} |\nabla v|_{g+g_e}^2 + s_g v^2 dv_{g+g_e}}{\int_{M \times \mathbb{R}^n} u^{p_{m+n}-2} v^2 dv_{g+g_e}} \left( \int_{M \times \mathbb{R}^n} u^{p_{m+n}} dv_{g+g_e} \right)^{\frac{2}{m+n}}.$$

is continous, then  $F$  depends continuously on  $u$  and  $V$ .

Let  $u \in C_c^\infty(M \times \mathbb{R}^n)$  be a non-negative function with support included in  $M \times B_R(0)$ . We claim that for any  $V \in Gr_u^2(C_c^\infty(M \times B_R(0)))$ ,

$$F(u, V) \geq 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]).$$

Without loss of generality we can assume that  $\|u\|_{p_{m+n}} = 1$ . Let  $k$  be a positive integer, we define

$$u_k(p, q) := \frac{u(p, q) + \frac{1}{k} \chi_{(M \times \overline{B_R(0)})}(p, q)}{\|u + \frac{1}{k} \chi_{(M \times \overline{B_R(0)})}\|_{p_{m+n}}}$$



where  $\chi_{(M \times \overline{B_R(0)})}$  is the characteristic function of  $M \times \overline{B_R(0)}$ .

We are going to proceed in a similar manner to the proof of Lemma 3.4. Let us consider the operator  $P_k : C_c^\infty(M \times B_R(0)) \longrightarrow C_c^\infty(M \times B_R(0))$  defined by

$$P_k(v) := a_{m+n} u_k^{\frac{2-p_{m+n}}{2}} \Delta_{g+g_e} (u_k^{\frac{2-p_{m+n}}{2}} v) + s_g u_k^{(2-p_{m+n})} v.$$

If  $\lambda_1^k \leq \lambda_2^k$  are the first two eigenvalues of the Dirichlet problem for  $P_k$  on  $M \times \overline{B_R(0)}$ , and  $v_1^k$  and  $v_2^k$  their respective associated eigenvectors, then  $u_k^{-\frac{p_{m+n}}{2}} v_1^k$  and  $u_k^{-\frac{p_{m+n}}{2}} v_2^k$  are eigenvectors of the conformal Laplacian  $L_{u_k^{p_{m+n}-2}(g+g_e)}$  with eigenvalues  $\lambda_1^k$  and  $\lambda_2^k$ , respectively. We can choose  $v_1^k$  and  $v_2^k$  such that for  $w_1 := u_k^{\frac{2-p_{m+n}}{2}} v_1^k$  and  $w_2 := u_k^{\frac{2-p_{m+n}}{2}} v_2^k$  we have

$$(8) \quad L_{g+g_e}(w_1) = \lambda_1^k u_k^{p_{m+n}-2} w_1,$$

$$(9) \quad L_{g+g_e}(w_2) = \lambda_2^k u_k^{p_{m+n}-2} w_2,$$

and

$$(10) \quad \int_{M \times \mathbb{R}^n} u_k^{p_{m+n}-2} w_1 w_2 dv_{g+g_e} = 0.$$

By the maximum principle,  $w_1$  has no zeros in  $M \times B_R(0)$ . Hence, we can assume that  $w_1 > 0$  in  $M \times B_R(0)$ . Therefore, by equation (10),  $w_2$  must change sign in  $M \times B_R(0)$ . Let,  $z_1 := a \max(0, w_2)$  and  $z_2 := b \max(0, -w_2)$ . We choose  $a, b \in \mathbb{R}_{>0}$  such that

$$(11) \quad \int_{M \times \mathbb{R}^n} u_k^{p_{m+n}-2} z_l^2 dv_{g+g_e} = 1,$$

for  $l = 1, 2$ .

Then, by the Hölder inequality, we have

$$\begin{aligned} 2 &= \int_{M \times \mathbb{R}^n} u_k^{p_{m+n}-2} z_1^2 dv_{g+g_e} + \int_{M \times \mathbb{R}^n} u_k^{p_{m+n}-2} z_2^2 dv_{g+g_e} \\ &\leq \left( \int_{\{w_2 \geq 0\}} u_k^{p_{m+n}} dv_{g+g_e} \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \left( \int_{M \times B_R(0)} z_1^{p_{m+n}} dv_{g+g_e} \right)^{\frac{2}{p_{m+n}}} \\ &\quad + \left( \int_{\{w_2 < 0\}} u_k^{p_{m+n}} dv_{g+g_e} \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \left( \int_{M \times B_R(0)} z_2^{p_{m+n}} dv_{g+g_e} \right)^{\frac{2}{p_{m+n}}}. \end{aligned}$$

By the definition of the Yamabe constant, we obtain

$$2Y(M \times B_R(0), [g + g_e]) \leq \left[ \left( \int_{\{w_2 \geq 0\}} u_k^{p_{m+n}} dv_{g+g_e} \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \right]$$

$$\begin{aligned} & \times \left( \int_{M \times B_R(0)} L_{g+g_e}(z_1) z_1 dv_{g+g_e} \right) \\ & + \left( \int_{\{w_2 < 0\}} u_k^{p_{m+n}} dv_{g+g_e} \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \left( \int_{M \times B_R(0)} L_{g+g_e}(z_2) z_2 dv_{g+g_e} \right) \Big]. \end{aligned}$$

From equations (8), (9), and (11) we get that

$$\begin{aligned} 2Y(M \times B_R(0), [g + g_e]) & \leq \lambda_2^k \left[ \left( \int_{\{w_2 \geq 0\}} u_k^{p_{m+n}} dv_{g+g_e} \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \right. \\ & \left. + \left( \int_{\{w_2 < 0\}} u_k^{p_{m+n}} dv_{g+g_e} \right)^{\frac{p_{m+n}-2}{p_{m+n}}} \right]. \end{aligned}$$

Then, applying again the Hölder inequality, we have that

$$2Y(M \times B_R(0), [g + g_e]) \leq \lambda_2^k 2^{\frac{2}{p_{m+n}}}.$$

Therefore,

$$2^{\frac{2}{m+n}} Y(M \times B_R(0), [g + g_e]) \leq \lambda_2^k.$$

Since  $\lambda_2^k = \inf_{V \in Gr_u^2(C_c^\infty(M \times B_R(0)))} F(u_k, V)$ , we have proved the claim for  $u_k$ . By the continuity of  $F$  with respect to the first variable, letting  $k$  go to infinity we obtain that

$$F(u, V) \geq 2^{\frac{2}{m+n}} Y(M \times B_R(0), [g + g_e]) \geq 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e])$$

for any  $V \in Gr_u^2(C_c^\infty(M \times B_R(0)))$ .

Therefore, for any  $u \in C_c^\infty(M \times \mathbb{R}^n)$  and  $V \in Gr_u^2(C_c^\infty(M \times \mathbb{R}^n))$ , we can choose  $R$  sufficiently large such that  $u \in C_c^\infty(M \times B_R(0))$  and  $V \in Gr_u^2(C_c^\infty(M \times B_R(0)))$ , and then we apply the claim.

Thereby, we obtain that

$$Y^2(M \times \mathbb{R}^n, g + g_e) \geq 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e]).$$

□

As a consequence of Theorem 1.4, we can rewrite the statements of Theorem 1.1 and Theorem 1.3 as follows:

**Theorem 3.9.** *Let  $(M^m, g)$  be a closed manifold ( $m \geq 2$ ) with positive scalar curvature and let  $(N^n, h)$  be any closed manifold. Then,*

$$\lim_{t \rightarrow +\infty} Y^2(M \times N, [g + th]) = Y^2(M \times \mathbb{R}^n, g + g_e).$$

*If in addition  $s_g$  is constant, then*

$$\lim_{t \rightarrow +\infty} Y_N^2(M \times N, g + th) = Y_{\mathbb{R}^n}^2(M \times \mathbb{R}^n, g + g_e).$$

4. SECOND YAMABE AND SECOND  $N$ -YAMABE INVARIANT

Throughout this section  $W^k$  will be a closed manifold of dimension  $k$ .

If  $Y(W, [G]) \geq 0$ , then  $Y(W, [G]) = Y^1(W, [G])$ . Therefore, we have that  $Y(W) = Y^1(W)$  if  $W$  admits a metric of constant scalar curvature equal to zero. Recall that if  $Y(W) > 0$ , then  $W$  admits such metrics (cf. [14]). By ([2], Proposition 8.1), we know that if  $Y^l(W, [G]) < 0$ , then  $Y^l(W, [G]) = -\infty$ . Hence, if  $Y(W) < 0$  or  $Y(W) = 0$  and the Yamabe invariant is not attained, then the first Yamabe invariant of  $W$  must be  $-\infty$ .

Note that the infimum of the  $l$ th Yamabe constant over the space of Riemannian metrics of  $W$  is always  $-\infty$ . Indeed, for every positive integer  $l$ , we can find a metric  $G$  such that the first  $l$  eigenvalues of  $L_G$  are negative (cf. [9], Proposition 3.2), which implies that  $Y^1(M, [G]) = \dots = Y^l(M, [G]) = -\infty$ .

## 4.1. Second Yamabe Invariant.

**Proposition 4.1.**  $Y^2(W) > -\infty$  if and only if  $Y^2(W) \geq 0$ .

*Proof.* Suppose that  $Y^2(W) < 0$ . Then, the second Yamabe constant of any metric  $G$  is negative, which implies that  $Y^2(W, [G]) = -\infty$ . Therefore,  $Y^2(W) = -\infty$ .  $\square$

**Lemma 4.2.** Let  $[G]$  be a conformal class of  $W$  and let  $\tilde{G} \in [G]$ . Then,  $\lambda_l(L_G)$  and  $\lambda_l(L_{\tilde{G}})$  have the same sign.

*Proof.* Let  $u \in C_{>0}^\infty(W)$  such that  $\tilde{G} = u^{p-2}G$ . Assume that  $\lambda_l(L_G) > 0$  and  $\lambda_l(L_{\tilde{G}}) \leq 0$ . Let  $V_0 \in Gr^l(H_1^2(W))$  that realizes  $\lambda_l(L_{\tilde{G}})$ . Then,

$$\sup_{v \in V_0 - \{0\}} \frac{\int_W v L_G(v) dv_G}{\int_W u^{p_k-2} v^2 dv_G} = \lambda_l(L_{\tilde{G}}) \leq 0,$$

which implies that  $\int_W v L_g(v) dv_G \leq 0$  for any  $v \in V_0 - \{0\}$ . Therefore, we obtain

$$0 < \lambda_l(L_G) \leq \sup_{v \in V_0 - \{0\}} \frac{\int_W v L_G(v) dv_G}{\int_W v^2 dv_G} \leq 0,$$

which is a contradiction. Hence,  $\lambda_l(L_{\tilde{G}}) > 0$ .

Now, assume that  $\lambda_l(L_G) = 0$ . Is easy to see that  $\lambda_l(L_{\tilde{G}})$  can not be negative. If  $\lambda_l(L_{\tilde{G}}) > 0$ , then we are in the same situation as above. Exchanging  $G$  by  $\tilde{G}$ , we get that  $\lambda_l(L_G) > 0$ , which is again a contradiction. Thus,  $\lambda_l(L_{\tilde{G}}) = 0$ .  $\square$

**Lemma 4.3.**  $Y^2(W) = -\infty$  if and only if the second eigenvalue of the conformal Laplacian is negative for all the Riemannian metrics on  $W$ .

*Proof.* If for any metric  $\lambda_2(L_G) < 0$ , then  $Y^2(W, [G]) = -\infty$ . Thus, if this is fulfilled for all the metrics on  $W$ , then  $Y^2(W) = -\infty$ .

Now assume that  $Y^2(W) = -\infty$ . Therefore, for any metric  $G$  we have

$$Y^2(W, G) = \inf_{h \in [G]} \lambda_l(L_h) \text{vol}(W, h)^{\frac{2}{k}} = -\infty.$$

Hence, there exists a metric  $\tilde{G}$  in the conformal class  $[G]$  with  $\lambda_2(L_{\tilde{G}}) < 0$ . By Lemma 4.2,  $\lambda_2(L_G)$  must be negative.  $\square$

**Proposition 4.4.** *If  $Y^2(W) = -\infty$ , then  $Y(W) \leq 0$ .*

*Proof.* Lemma 4.3 implies that  $\lambda_2(L_G) < 0$  for any metric  $G$  on  $W$ . Therefore, the first eigenvalue of  $L_G$  is negative, and consequently  $Y(W, [G]) < 0$ . Thereby,  $Y(W) \leq 0$ .  $\square$

**Example 4.5.**

a) Let  $M$  be a closed manifold with  $Y(M) < 0$ . For instance, take  $M = \mathbb{H}^3/\Gamma$  any compact quotient of the 3-dimensional Hyperbolic space. Let us consider  $W := M \sqcup M$ , the disjoint union of two copies of  $M$ . We denote with  $M_i$  ( $i = 1, 2$ ) the copies of  $M$ . If  $G$  is any metric on  $W$ , let us denote by  $G_i$  the restriction of  $G$  to  $M_i$ . Recall as the sign of the first eigenvalue of the conformal Laplacian has the same sign that the Yamabe constant. Thereby,

$$\lambda_2(L_G) = \min \left( \max_{i=1,2} (\lambda_1(L_{G_i})), \lambda_2(L_{G_1}), \lambda_2(L_{G_2}) \right) < 0$$

and

$$Y^2(M \sqcup M) = -\infty.$$

b) Let  $M$  be a compact quotient of a non abelian nilpotent Lie group. It is known that  $Y(M) = 0$  but the Yamabe invariant is not attained by any conformal class. Then,  $W = M \sqcup M$  satisfies that  $Y^2(W) = -\infty$  and  $Y(W) = 0$ .

**Proposition 4.6.** *If  $W$  admits a metric of zero scalar curvature, then  $Y^2(W) > 0$ .*

*Proof.* If  $Y(W) > 0$ , then it is clear that  $Y^2(W) > 0$ . Assume that  $Y(W) = Y(W, [G_0]) = 0$  for some metric  $G_0$ . Then,  $\lambda_1(L_{G_0}) = 0$  and  $\lambda_2(L_{G_0}) > 0$ . Therefore,  $Y^2(W, [G_0]) \geq 0$ . If  $Y^2(W, [G_0]) > 0$ , then we have nothing to prove. If  $Y^2(W, [G_0]) = 0$ , then by Theorem 2.2 part a) the second Yamabe constant is achieved by a generalized metric  $\tilde{G}$ . Therefore  $\lambda_2(L_{\tilde{G}}) = 0$ , which is a contradiction.  $\square$

**Remark 4.7.** Let  $N$  be a closed manifold obtained by performing surgery on  $(W, G)$  of codimension at least 3. Bär and Dahl proved in ([6], Theorem 3.1) that given  $l \in \mathbb{N}$  and  $\varepsilon > 0$  there exists a metric  $H$  on  $N$  such that  $|\lambda_i(L_H) - \lambda_i(L_G)| < \varepsilon$  for  $1 \leq i \leq l$ . Therefore, the positivity of the second Yamabe invariant is preserved under surgery of codimension at least 3.

#### 4.2. Bounds for the second Yamabe invariant and the second $N$ -Yamabe invariant.

An immediate consequence of the Theorem 2.1 is the following proposition:

**Proposition 4.8.** *If  $W$  admits a metric of zero scalar curvature, then*

$$2^{\frac{2}{k}} Y(W) \leq Y^2(W) \leq [Y(W)^{\frac{k}{2}} + Y(S^k)^{\frac{k}{2}}]^{\frac{2}{k}}.$$

If  $W = S^k$ ,  $Y^2(S^k) = 2^{\frac{2}{k}}Y(S^k)$ . From Theorem 2.1, we have that  $Y^2(S^k, [g_0^k]) = 2^{\frac{2}{k}}Y(S^k)$ . Hence, the second Yamabe invariant of  $S^k$  is achieved by the second Yamabe constant of the conformal class  $[g_0^k]$ . But recall that  $Y^2(S^k, [g_0^k])$  is not achieved, even by a generalized metric.

Also, it follows from Proposition 4.8 that the second Yamabe invariant of a  $k$ -dimensional manifold is bounded from above by the second Yamabe invariant of the  $k$ -dimensional sphere:

$$Y^2(W) \leq Y^2(S^k).$$

**Example 4.9.** Let  $G$  be the Riemannian metric on  $S^k \sqcup S^k$  whose restriction to each copy of  $S^k$  is  $g_0^k$ . Then,  $Y^2(S^k \sqcup S^k, G) = 2^{\frac{2}{k}}Y(S^k)$  ( cf. [2], Proposition 5.1). Thus,  $Y^2(S^k \sqcup S^k) = Y^2(S^k)$ .

**Example 4.10.** Let  $W = S^{k-1} \times S^1$  ( $k \geq 3$ ). Using that  $Y(S^{k-1} \times S^1) = Y(S^k)$  ( cf. [15] and [23]) it follows from Proposition 4.8 that  $Y^2(S^{k-1} \times S^1) = 2^{\frac{2}{k}}Y(S^k)$ .

**Example 4.11.** It was computed by LeBrun in [16] that  $Y(\mathbb{CP}^2) = 12\sqrt{2}\pi$ . Then,  $24\pi \leq Y^2(\mathbb{CP}^2) \leq 4\sqrt{42}\pi$ .

Bray and Neves proved in [7] that  $Y(\mathbb{RP}^3) = 2^{-\frac{2}{3}}Y(S^3)$ . Therefore, the second Yamabe invariant of  $\mathbb{RP}^3$  is bounded by  $Y(S^3) \leq Y^2(\mathbb{RP}^3) \leq (\frac{3}{2})^{\frac{2}{3}}Y(S^3)$ .

Both, are examples where the second Yamabe invariant is positive but strictly minor than the second Yamabe invariant of the sphere.

Let  $M^m$  and  $N^n$  be closed manifolds ( $m, n \geq 2$ ) with positive Yamabe invariant. An immediate consequence of Theorem 1.1 is that

$$Y^2(M \times N) \geq 2^{\frac{2}{m+n}} \sup_{\{s_g > 0, s_h > 0\}} \max(Y(M \times \mathbb{R}^n, [g + g_e^n]), Y(N \times \mathbb{R}^m, [h + g_e^m])).$$

For  $S^n \times S^n$ , we get that  $Y^2(S^n \times S^n) \geq 2^{\frac{1}{n}}Y(S^n \times \mathbb{R}^n, [g_0^n + g_e^n])$ . Note that if  $Y(M) > 0$  and  $N$  is any closed manifold, then  $Y^2(M \times N) > 0$ .

In the following proposition we use several known lower bounds for the Yamabe invariant to deduce lower bounds for the second Yamabe invariant of a Riemannian product.

**Proposition 4.12.**

i) Let  $M^m \times N^n$  with  $m, n \geq 3$  and  $Y(M) > 0$ . Then,

$$Y^2(M \times N) \geq 2^{\frac{2}{m+n}} B_{m,n} Y(M)^{\frac{m}{m+n}} Y(S^n)^{\frac{n}{m+n}}.$$

where  $B_{m,n} = a_{m+n}(m+n)(ma_m)^{-\frac{m}{m+n}}(na_n)^{-\frac{n}{m+n}}$ .

ii) Let  $M$  be a 2-dimensional closed manifold. Then,

$$Y^2(M \times S^2) \geq \frac{2c}{3^{\frac{3}{4}}}Y(S^4),$$

where  $c = (1.047)^2$ .

- iii) Let  $(M^m, g)$  be a closed manifold with Ricci curvature bounded from below by  $(m-1)$ . Then,

$$Y^2(M \times S^1) \geq 2^{\frac{2}{m+1}} \left( \frac{\text{vol}(M, g)}{\text{vol}(S^m, g_0^m)} \right)^{\frac{2}{m+1}} Y(S^{m+1}).$$

- iv) Let  $M^3$  and  $N^2$  be closed manifolds. Then,

$$Y^2(M \times S^2) \geq 2^{\frac{2}{5}} (0.62) Y(S^5)$$

and

$$Y^2(N \times S^3) \geq 2^{\frac{2}{5}} (0.75) Y(S^5).$$

The statements in Proposition 4.12 are immediate consequence of apply Proposition 4.8 to the lower bounds for the Yamabe invariant obtained in [4], [17], [19], and [20]. In all the cases, in order to obtain the bounds, Theorem 2.3 (first equality) is used. In [19] and [17], the authors estimated the isoperimetric profile of  $S^2 \times \mathbb{R}^2$  and  $M \times S^1$  and used them to obtain lower bounds for  $Y(M \times \mathbb{R}^2)$  and  $Y(M \times \mathbb{R})$  respectively. In [20], the authors compare the isoperimetric profile of  $S^2 \times \mathbb{R}^3$  and  $S^3 \times \mathbb{R}^2$  with the one of  $S^5$ , and used it to obtain a lower bounds of  $Y(S^2 \times \mathbb{R}^3, [g_0^2 + g_e])$  and  $Y(S^3 \times \mathbb{R}^2, [g_0^3 + g_e])$ . In the following, for convenience of the reader, we state the bounds obtained by Ammann, Dahl, and Humbert, Petean, and Petean and Ruiz:

- i) In [4], Ammann, Dahl, and Humbert proved that the Yamabe invariant of a Riemannian product  $M^m \times N^n$  with  $m, n \geq 3$  and  $Y(M) \geq 0$  is bounded from below by

$$Y(M \times N) \geq B_{m,n} Y(M)^{\frac{m}{m+n}} Y(N)^{\frac{n}{m+n}}.$$

- ii) In [19], Petean and Ruiz proved that for any 2-dimensional manifold  $M$

$$Y(M \times S^2) \geq \frac{\sqrt{2}c}{3^{\frac{3}{4}}} Y(S^4).$$

- iii) It was proved by Petean in [17] that if  $(M^m, g)$  is a closed Riemannian manifold with  $\text{Ricci}(g) \geq (m-1)g$ , then

$$Y(M \times \mathbb{R}, [g + g_e]) \geq \left( \frac{\text{vol}(M, g)}{\text{vol}(S^m, g_0^m)} \right)^{\frac{2}{m+1}} Y(S^{m+1}).$$

- iv) In [20], Petean and Ruiz proved that if  $M$  is a closed 3-dimensional manifold and if  $N$  is a closed 2-dimensional manifold, then  $Y(M \times S^2) \geq 0.63Y(S^5)$  and  $Y(N \times S^3) \geq 0.75Y(S^5)$ .

**Proposition 4.13.** *Let  $M^m$  be a closed manifold with  $Y(M) > 0$  and  $N^n$  any closed manifold. Then,*

$$Y_N^2(M \times N) \geq \frac{2^{\frac{2}{m+n}} A_{m,n} Y(M)^{\frac{m}{m+n}}}{\alpha_{m,n}}.$$

*Proof.* Let  $g$  be a Yamabe metric with positive Yamabe constant and unit volume. Let  $h$  be any Riemannian metric on  $N$ . From Theorem 1.3 and Corollary 3.6 we obtain

$$\begin{aligned} Y_N^2(M \times N) &\geq \lim_{t \rightarrow +\infty} Y_N^2(M \times N, g + th) \\ &= 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e) = \frac{2^{\frac{2}{m+n}} A_{m,n} Y(M, [g])^{\frac{m}{m+n}}}{\alpha_{m,n}}. \end{aligned}$$

The proposition follows taking the supreme over the set of Yamabe metrics on  $M$  with unit volume.  $\square$

**Example 4.14.** From the proposition above we get that  $Y_{S^2}^2(S^2 \times S^2) \geq 84.01080$  and  $Y_{S^3}^3(S^3 \times S^3) \geq 119.33249$ . Here, we used the numerical computations of the Glariardo-Nirenberg constants carried out in [1], i.e,  $\alpha_{2,2} = 0.41343$  and  $\alpha_{3,3} = 0.31257$ .

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